BLM REALIZATION FOR THE INTEGRAL FORM OF QUANTUM $\mathfrak{gl}_n$

QIANG FU

ABSTRACT. Let $\mathbf{U}(n)$ be the quantum enveloping algebra of $\mathfrak{gl}_n$ over $\mathbb{Q}(v)$, where v is an indeterminate. We will use q-Schur algebras to realize the integral form of $\mathbf{U}(n)$. Furthermore we will use this result to realize quantum $\mathfrak{gl}_n$ over k, where k is a field containing an l-th primitive root ε of 1 with $l \geqslant 1$ odd.

1. Introduction

It is well known that the positive part of the integral form of quantum enveloping algebras of finite type was realized as a Ringel–Hall algebra (see [16, 17]). Using a beautiful geometric construction of q-Schur algebras, the entire quantum $\mathfrak{gl}_n$ over the rational function field $\mathbb{Q}(v)$ (with v being an indeterminant) was realized by A. A. Beilinson, G. Lusztig and R. MacPherson (BLM) in [1].

Let $U(n)$ be the Lusztig $\mathcal{Z}$-form of quantum $\mathfrak{gl}_n$, where $\mathcal{Z} = \mathbb{Z}[v, v^{-1}]$. We will give BLM realization of $U(n)$ in this paper. More precisely, We will construct a certain $\mathcal{Z}$-submodule of $\prod_{r\geqslant 0} \boldsymbol{\mathcal{S}}(n,r)$, denoted by $\mathcal{V}(n)$, where $\boldsymbol{\mathcal{S}}(n,r)$ is the q-Schur algebra over $\mathbb{Q}(v)$. We will show that $\mathcal{V}(n)$ is a $\mathcal{Z}$-subalgebra of $\prod_{r\geqslant 0} \boldsymbol{\mathcal{S}}(n,r)$ and prove in 4.4 that $\mathcal{V}(n)$ is isomorphic to $U(n)$ as a $\mathcal{Z}$-algebra. Similarly, we may construct the affine version of $\mathcal{V}(n)$, denoted by $\mathcal{V}_\vartriangle(n)$, which is a certain $\mathcal{Z}$-submodule of $\prod_{r\geqslant 0} \boldsymbol{\mathcal{S}}_\vartriangle(n,r)$, where $\boldsymbol{\mathcal{S}}_\vartriangle(n,r)$ is the affine q-Schur algebra over $\mathbb{Q}(v)$. We conjecture that $\mathcal{V}_\vartriangle(n)$ is a $\mathcal{Z}$-subalgebra of $\prod_{r\geqslant 0} \boldsymbol{\mathcal{S}}_\vartriangle(n,r)$. If this conjecture is true, then $\mathcal{V}_\vartriangle(n)$ is isomorphic to the $\mathcal{Z}$-module $\widetilde{\mathfrak{D}}_\vartriangle(n)$ defined in [2, (3.8.1.1)].

Let k be a field containing an l-th primitive root ε of 1 with $l \geqslant 1$ odd. Specializing v to ε, k will be viewed as a $\mathcal{Z}$-module. Let $U_k(n) = U(n) \otimes_{\mathcal{Z}} k$ and $\overline{U_k(n)} = U_k(n)/\langle K_i^l - 1 \mid 1 \leqslant i \leqslant n-1\rangle$. We will prove that the algebra $\overline{U_k(n)}$ can be realized as a k-subalgebra of $\prod_{r\geqslant 0} \mathcal{S}_k(n,r)$, where $\mathcal{S}_k(n,r)$ is the q-Schur algebra over k.

We organize this paper as follows. We recall some results of quantum $\mathfrak{gl}_n$ and q-Schur algebras in §2. We will establish some useful multiplication formulas for q-Schur algebras in 3.4 and 3.5. A certain $\mathcal{Z}$-submodule of $\prod_{r\geqslant 0} \boldsymbol{\mathcal{S}}(n,r)$, denoted by $\mathcal{V}(n)$, will be constructed in §4. We will use 3.4 and 3.5 to prove that $\mathcal{V}(n)$ is BLM realization of $U(n)$. Furthermore, we will give realization of $\overline{U_k(n)}$ in 4.6.

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Throughout this paper, let $\mathcal{Z}=\mathbb{Z}[v,v^{-1}]$, where v is an indeterminate, and let $\mathbb{Q}(v)$ be the fraction field of $\mathcal{Z}$. For $i\in\mathbb{Z}$ let $[i]=\frac{v^i-v^{-i}}{v-v^{-1}}$ and $[\![i]\!]=\frac{v^{2i}-1}{v^2-1}$. For integers N,t with $t\geqslant 0$, let

$$\begin{bmatrix}N\\t\end{bmatrix}=\frac{[N][N-1]\cdots[N-t+1]}{[t]^!}\in\mathcal{Z},\quad \left[\!\!\left[\begin{matrix}N\\t\end{matrix}\right]\!\!\right]=\frac{[\![N]\!][\![N-1]\!]\cdots[\![N-t+1]\!]}{[\![t]\!]^!}\in\mathcal{Z}$$

where $[t]^!=[1][2]\cdots[t]$ and $[\![t]\!]^!=[\![1]\!][\![2]\!]\cdots[\![t]\!]$. For $\mu\in\mathbb{Z}^n$ and $\lambda\in\mathbb{N}^n$ let $\left[{\mu\atop\lambda}\right]=\left[{\mu_1\atop\lambda_1}\right]\cdots\left[{\mu_n\atop\lambda_n}\right]$.

2. The quantum $\mathfrak{gl}_n$ and the q-Schur algebra

The below definition of quantum $\mathfrak{gl}_n$ is a slightly modified version of Jimbo [11]; see [9, 18].

Definition 2.1. The quantum enveloping algebra of $\mathfrak{gl}_n$ is the $\mathbb{Q}(v)$-algebra $\mathbf{U}(n)$ presented by generators

$$E_i,\ F_i\quad(1\leqslant i\leqslant n-1),\ K_j,\ K_j^{-1}\quad(1\leqslant j\leqslant n)$$

and relations

(a) $K_iK_j=K_jK_i,\ K_iK_i^{-1}=1$;
(b) $K_iE_j=v^{\delta_{i,j}-\delta_{i,j+1}}E_jK_i$;
(c) $K_iF_j=v^{\delta_{i,j+1}-\delta_{i,j}}F_jK_i$;
(d) $E_iE_j=E_jE_i,\ F_iF_j=F_jF_i$ *when* $|i-j|>1$;
(e) $E_iF_j-F_jE_i=\delta_{i,j}\frac{\widetilde{K}_i-\widetilde{K}_i^{-1}}{v-v^{-1}}$, *where* $\widetilde{K}_i=K_iK_{i+1}^{-1}$;
(f) $E_i^2E_j-(v+v^{-1})E_iE_jE_i+E_jE_i^2=0$ *when* $|i-j|=1$;
(g) $F_i^2F_j-(v+v^{-1})F_iF_jF_i+F_jF_i^2=0$ *when* $|i-j|=1$.

Following [13], let $U(n)$ be the $\mathcal{Z}$-subalgebra of $\mathbf{U}(n)$ generated by all $E_i^{(m)}$, $F_i^{(m)}$, $K_i^{\pm1}$ and $\left[{K_i;0\atop t}\right]$, where for $m,t\in\mathbb{N}$,

$$E_i^{(m)}=\frac{E_i^m}{[m]^!},\ F_i^{(m)}=\frac{F_i^m}{[m]^!},\text{ and }\begin{bmatrix}K_i;0\\t\end{bmatrix}=\prod_{s=1}^t\frac{K_iv^{-s+1}-K_i^{-1}v^{s-1}}{v^s-v^{-s}}.$$

Let $\Theta(n)$ be the set of all $n\times n$ matrices over $\mathbb{N}$. Let $\Theta^\pm(n)$ be the set of all $A\in\Theta(n)$ whose diagonal entries are zero. Let $\Theta^+(n)$ (resp., $\Theta^-(n)$) be the subset of $\Theta(n)$ consisting of those matrices A with $a_{i,j}=0$ for all $i>j$ (resp., $i<j$). For $A\in\Theta^\pm(n)$, write $A=A^++A^-$ with $A^+\in\Theta^+(n)$ and $A^-\in\Theta^-(n)$. For $A\in\Theta^\pm(n)$ let

$$E^{(A^+)}=\prod_{1\leqslant i\leqslant h<j\leqslant n}E_h^{(a_{i,j})}\text{ and }F^{(A^-)}=\prod_{1\leqslant j\leqslant h<i\leqslant n}F_h^{(a_{i,j})}\tag{2.1.1}$$

The orders in which the products $E^{(A^+)}$ and $F^{(A^-)}$ are taken are defined as follows. Put

$$M_j=M_j(A^+)=E_{j-1}^{(a_{j-1,j})}(E_{j-2}^{(a_{j-2,j})}E_{j-1}^{(a_{j-2,j})})\cdots(E_1^{(a_{1,j})}E_2^{(a_{1,j})}\cdots E_{j-1}^{(a_{1,j})}).$$

Similarly, put

$$M_j'=(F_{j-1}^{(a_{j,1})}\cdots F_2^{(a_{j,1})}F_1^{(a_{j,1})})\cdots(F_{j-1}^{(a_{j,j-2})}F_{j-2}^{(a_{j,j-2})})F_{j-1}^{(a_{j,j-1})}.$$

Then $E^{(A^+)} = M_n M_{n-1} \cdots M_2$ and $F^{(A^-)} = M_2' M_3' \cdots M_n'$. According to [13, 4.5] and [14, 7.8] we have the following result.

Proposition 2.2. *The set*

$$\{E^{(A^+)} \prod_{1 \leqslant i \leqslant n} K_i^{\delta_i} \begin{bmatrix} K_i; 0 \\ \lambda_i \end{bmatrix} F^{(A^-)} \mid A \in \Theta^{\pm}(n),\, \delta, \lambda \in \mathbb{N}^n,\, \delta_i \in \{0,1\},\, \forall i\}$$

forms a $\mathcal{Z}$-basis of $U(n)$.

Schur algebras are certain important finite-dimensional algebras. It is used to link representation of general linear groups and symmetric groups. q-Schur algebras are quantum deformation of Schur algebras, which is defined by certain endomorphism algebras arising from Hecke algebras of type A. We now follow [3, 4] to recall the definition of q-Schur algebras as follows. Let $\mathfrak{S}_r$ be the symmetric group on r letters. The symmetric group $\mathfrak{S}_r$ is generated by the set $\{s_i := (i, i+1) \mid 1 \leqslant i \leqslant r-1\}$. The Hecke algebra $\mathcal{H}(r)$ associated with $\mathfrak{S}_r$ is the $\mathcal{Z}$-algebra generated by T_i $(1 \leqslant i \leqslant r-1)$, with the following relations:

$$(T_i + 1)(T_i - q) = 0, \;\; T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}, \;\; T_i T_j = T_j T_i \; (|i-j| > 1).$$

where $q = v^2$. If $w = s_{i_1} s_{i_2} \cdots s_{i_m}$ is reduced let $T_w = T_{i_1} T_{i_2} \cdots T_{i_m}$. Then the set $\{T_w \mid w \in \mathfrak{S}_r\}$ forms a $\mathcal{Z}$-basis for $\mathcal{H}(r)$. Let $\Lambda(n,r) = \{\lambda \in \mathbb{N}^n \mid \sigma(\lambda) := \sum_{1 \leqslant i \leqslant n} \lambda_i = r\}$. For $\lambda \in \Lambda(n,r)$, let $\mathfrak{S}_\lambda$ be the Young subgroup of $\mathfrak{S}_r$ and let $x_\lambda = \sum_{w \in \mathfrak{S}_\lambda} T_w$. Let $\boldsymbol{\mathcal{H}}(r) = \mathcal{H}(r) \otimes_{\mathcal{Z}} \mathbb{Q}(v)$. The endomorphism algebras

$$\mathcal{S}(n,r) := \mathrm{End}_{\mathcal{H}(r)}\Big(\bigoplus_{\lambda \in \Lambda(n,r)} x_\lambda \mathcal{H}(r)\Big), \quad \boldsymbol{\mathcal{S}}(n,r) := \mathrm{End}_{\boldsymbol{\mathcal{H}}(r)}\Big(\bigoplus_{\lambda \in \Lambda(n,r)} x_\lambda \boldsymbol{\mathcal{H}}(r)\Big)$$

are called q-Schur algebras. For $\lambda, \mu \in \Lambda(n,r)$ let $\mathscr{D}_{\lambda,\mu}$ be the set of distinguished double $(\mathfrak{S}_\lambda, \mathfrak{S}_\mu)$-coset representatives. For $\lambda, \mu \in \Lambda(\eta, r)$, $d \in \mathscr{D}_{\lambda,\mu}$, define $\phi^d_{\lambda\mu} \in \mathcal{S}(n,r)$ by

$$\phi^d_{\lambda\mu}(x_\nu h) = \delta_{\mu,\nu} \sum_{x \in \mathfrak{S}_\lambda d \mathfrak{S}_\mu} T_x h.$$

According to [4, 1.4], the set $\{\phi^d_{\lambda\mu} \mid \lambda, \mu \in \Lambda(n,r),\, d \in \mathscr{D}_{\lambda,\mu}\}$ forms a $\mathcal{Z}$-basis for $\mathcal{S}(n,r)$.

Let $\Theta(n,r) = \big\{A \in \Theta(n) \,\big|\, \sigma(A) := \sum_{1 \leqslant i,j \leqslant n} a_{i,j} = r\big\}$. The basis for $\mathcal{S}(n,r)$ can also be indexed by the set $\Theta(n,r)$, which we now describe. For $1 \leqslant i \leqslant n$, and $\lambda \in \Lambda(n,r)$ let

$$R_i^\lambda = \Big\{\sum_{1 \leqslant t \leqslant i-1} \lambda_t + 1, \sum_{1 \leqslant t \leqslant i-1} \lambda_t + 2, \ldots, \sum_{1 \leqslant t \leqslant i-1} \lambda_t + \lambda_i\Big\},$$

According to [10, 1.3.10], there is a bijective map

$$\jmath : \{(\lambda, d, \mu) \mid d \in \mathscr{D}_{\lambda,\mu}, \lambda, \mu \in \Lambda(n,r)\} \longrightarrow \Theta(n,r)$$

sending (λ, d, μ) to $A = (a_{k,l})$, where $a_{k,l} = |R_k^\lambda \cap dR_l^\mu|$ for all $k, l \in \mathbb{Z}$. If $\lambda, \mu \in \Lambda(n, r)$ and $d \in \mathscr{D}_{\lambda,\mu}$ are such that $A = \jmath(\lambda, d, \mu)$, let

$$[A] = v^{-d_A} \phi_{\lambda,\mu}^d, \quad \text{where} \quad d_A = \sum_{\substack{1 \leqslant i \leqslant n \\ i \geqslant k, j < l}} a_{i,j} a_{k,l}.$$

Then the set $\{[A] \mid A \in \Theta(n, r)\}$ forms a $\mathcal{Z}$-basis for $\mathcal{S}(n, r)$.

The geometric definition of q-Schur algebra was given in [1, 1.2]. It is proved in [5, A.1] that the two definitions of q-Schur algebras are equivalent. According to [1, 1.2,1.3], for $\lambda \in \Lambda(n, r)$ and $A \in \Theta(n, r)$, we have

$$(2.2.1) \qquad [\mathrm{diag}(\lambda)][A] = \begin{cases} [A] & \text{if } \lambda = ro(A) \\ 0 & \text{otherwise;} \end{cases} \quad \text{and} \quad [A][\mathrm{diag}(\lambda)] = \begin{cases} [A] & \text{if } \lambda = co(A) \\ 0 & \text{otherwise,} \end{cases}$$

where $\mathrm{ro}(A) = (\sum_j a_{1,j}, \cdots, \sum_j a_{n,j})$ and $\mathrm{co}(A) = (\sum_i a_{i,1}, \cdots, \sum_i a_{i,n})$ are the sequences of row and column sums of A.

The algebra $\mathbf{U}(n)$ and the q-Schur algebra $\boldsymbol{\mathcal{S}}(n, r)$ are related by an algebra epimorphism ζ_r which we now describe. For $A \in \Theta^\pm(n)$, $\delta \in \mathbb{Z}^n$ and $\lambda \in \mathbb{N}^n$ let

$$A(\delta, \lambda, r) = \sum_{\mu \in \Lambda(n, r - \sigma(A))} v^{\mu \centerdot \delta} \begin{bmatrix} \mu \\ \lambda \end{bmatrix} [A + \mathrm{diag}(\mu)] \in \boldsymbol{\mathcal{S}}(n, r);$$

$$A(\delta, r) = \sum_{\mu \in \Lambda(n, r - \sigma(A))} v^{\mu \centerdot \delta} [A + \mathrm{diag}(\mu)] \in \boldsymbol{\mathcal{S}}(n, r),$$

where $\mu \centerdot \delta = \sum_{1 \leqslant i \leqslant n} \mu_i \delta_i$. Furthermore we set

$$A(\delta, \lambda) = (A(\delta, \lambda, r))_{r \geqslant 0} \in \prod_{r \geqslant 0} \boldsymbol{\mathcal{S}}(n, r);$$

$$A(\delta) = (A(\delta, r))_{r \geqslant 0} \in \prod_{r \geqslant 0} \boldsymbol{\mathcal{S}}(n, r).$$

Then by definition we have $A(\delta) = A(\delta, \mathbf{0})$, where $\mathbf{0} = (0, \cdots, 0) \in \mathbb{N}^n$. For $1 \leqslant i, j \leqslant n$, let $E_{i,j} \in \Theta(n)$ be the matrix $(a_{k,l})$ with $a_{k,l} = \delta_{i,k}\delta_{j,l}$. According to [1], there is an algebra epimorphism

$$\zeta_r : \mathbf{U}(n) \twoheadrightarrow \boldsymbol{\mathcal{S}}(n, r)$$

satisfying $\zeta_r(E_h) = E_{h,h+1}(\mathbf{0}, r)$, $\zeta_r(K_1^{j_1} K_2^{j_2} \cdots K_n^{j_n}) = 0(\mathbf{j}, r)$ and $\zeta_r(F_h) = E_{h+1,h}(\mathbf{0}, r)$, for $1 \leqslant h \leqslant n-1$ and $\mathbf{j} \in \mathbb{Z}^n$.

We conclude this section by recalling an important triangular relation in q-Schur algebras. For $A = (a_{s,t}) \in \Theta(n)$ and $i < j$, let $\sigma_{i,j}(A) = \sum_{s \leqslant i; t \geqslant j} a_{s,t}$ and $\sigma_{j,i}(A) = \sum_{s \leqslant i; t \geqslant j} a_{t,s}$. Define $A' \preccurlyeq A$ iff $\sigma_{i,j}(A') \leqslant \sigma_{i,j}(A)$ and $\sigma_{j,i}(A') \leqslant \sigma_{j,i}(A)$ for all $1 \leqslant i < j \leqslant n$. Put $A' \prec A$ if $A' \preccurlyeq A$ and, for some pair (i, j) with $i \neq j$, $\sigma_{i,j}(A') < \sigma_{i,j}(A)$. According to [1, 5.3 and 5.4(c)], we have the following result.

Proposition 2.3. *For $A \in \Theta^{\pm}(n)$, we have*

$$\prod_{1\leqslant i\leqslant h<j\leqslant n}(a_{i,j}E_{h,h+1})(\mathbf{0})\cdot\prod_{1\leqslant j\leqslant h<i\leqslant n}(a_{i,j}E_{h+1,h})(\mathbf{0})=A(\mathbf{0})+f$$

where the ordering of the products in the left hand side of the above equation is the same as in (2.1.1) *and f is the $\mathbb{Q}(v)$-linear combination of $B(\mathbf{j})$ with $B\in\Theta^{\pm}(n)$, $B\prec A$ and $\mathbf{j}\in\mathbb{Z}^n$.*

3. The multiplication formulas for q-Schur algebras

We will derive certain useful multiplication formulas for q-Schur algebras in 3.4 and 3.5.

We need some preparation before proving 3.4 and 3.5. Let $^{-}:\mathcal{Z}\to\mathcal{Z}$ be the ring homomorphism defined by $\bar{v}=v^{-1}$. The following impotent multiplication formulas for q-Schur algebras was proved in [1, 3.4].

Proposition 3.1. *Let $1\leqslant h\leqslant n-1$, $A\in\Theta(n,r)$ and $\lambda=\mathrm{ro}(A)$. Let $B_m=\mathrm{diag}(\lambda)+mE_{h,h+1}-mE_{h+1,h+1}$ and $C_m=\mathrm{diag}(\lambda)-mE_{h,h}+mE_{h+1,h}$. Then in $\mathcal{S}(n,r)$*

(1) $[B_m]\cdot[A]=\displaystyle\sum_{\substack{\mathbf{t}\in\Lambda(n,m)\\ \forall u\in\mathbb{Z},t_u\leqslant a_{h+1,u}}}v^{\beta(\mathbf{t},A)}\overline{\prod_{u\in\mathbb{Z}}\left[\!\!\left[\begin{matrix}a_{h,u}+t_u\\ t_u\end{matrix}\right]\!\!\right]}\left[A+\sum_{u\in\mathbb{Z}}t_u(E^{\triangle}_{h,u}-E^{\triangle}_{h+1,u})\right]$;

for all $0\leqslant m\leqslant\lambda_{h+1}$, where $\beta(\mathbf{t},A)=\sum_{j\geqslant u}a_{h,j}t_u-\sum_{j>u}a_{h+1,j}t_u+\sum_{u<u'}t_ut_{u'}$.

(2) $[C_m]\cdot[A]=\displaystyle\sum_{\substack{\mathbf{t}\in\Lambda(n,m)\\ \forall u\in\mathbb{Z},t_u\leqslant a_{h,u}}}v^{\gamma(\mathbf{t},A)}\overline{\prod_{u\in\mathbb{Z}}\left[\!\!\left[\begin{matrix}a_{h+1,u}+t_u\\ t_u\end{matrix}\right]\!\!\right]}\left[A-\sum_{u\in\mathbb{Z}}t_u(E^{\triangle}_{h,u}-E^{\triangle}_{h+1,u})\right]$,

for all $0\leqslant m\leqslant\lambda_h$, where $\gamma(\mathbf{t},A)=\sum_{j\leqslant u}a_{h+1,j}t_u-\sum_{j<u}a_{h,j}t_u+\sum_{u<u'}t_ut_{u'}$.

We also need the following formulas for Gaussian binomial coefficient (see [12]).

Lemma 3.2. *For $m,n\in\mathbb{Z}$, $a,b\in\mathbb{N}$ we have*

(1) $\left[\!\!\left[{n\atop a}\right]\!\!\right]=\sum\limits_{0\leqslant j\leqslant a}v^{2(m-j)(a-j)}\left[\!\!\left[{m\atop j}\right]\!\!\right]\left[\!\!\left[{n-m\atop a-j}\right]\!\!\right]$;

(2) $\left[\!\!\left[{m\atop a}\right]\!\!\right]\left[\!\!\left[{m\atop b}\right]\!\!\right]=\sum\limits_{0\leqslant c\leqslant\min\{a,b\}}v^{2(b-c)(a-c)}\left[\!\!\left[{m\atop a+b-c}\right]\!\!\right]\left[\!\!\left[{a+b-c\atop c,\,a-c,\,b-c}\right]\!\!\right]$, *where* $\left[\!\!\left[{a+b-c\atop c,\,a-c,\,b-c}\right]\!\!\right]=\frac{[\![a+b-c]\!]^!}{[\![c]\!]^![\![a-c]\!]^![\![b-c]\!]^!}$.

Let $\leqslant$ be the partial order on $\mathbb{N}^n$ defined by setting, for $\lambda,\mu\in\mathbb{N}^n$, $\lambda\leqslant\mu$ if and only if $\lambda_i\leqslant\mu_i$ for $1\leqslant i\leqslant n$. For $\lambda,\alpha,\beta,\gamma\in\mathbb{N}^n$ with $\lambda=\alpha+\beta+\gamma$ let

$$\left[\begin{matrix}\lambda\\ \alpha,\ \beta,\ \gamma\end{matrix}\right]=\prod_{1\leqslant i\leqslant n}\frac{[\lambda_i]^!}{[\alpha_i]^![\beta_i]^![\gamma_i]^!}.$$

The above lemma immediately yields the following corollary.

Corollary 3.3. *For $\lambda,\mu\in\mathbb{N}^n$ and $\alpha,\beta\in\mathbb{Z}^n$ we have*

(1) $\left[{\alpha+\beta\atop\lambda}\right]=\sum\limits_{\mu\in\mathbb{N}^n,\,\mu\leqslant\lambda}v^{\alpha\centerdot(\lambda-\mu)-\mu\centerdot\beta}\left[{\alpha\atop\mu}\right]\left[{\beta\atop\lambda-\mu}\right]$;

(2) $\left[{\alpha\atop\lambda}\right]\left[{\alpha\atop\mu}\right]=\sum\limits_{\substack{\gamma\in\mathbb{N}^n\\ \gamma\leqslant\lambda,\,\gamma\leqslant\mu}}v^{\lambda\centerdot\mu-\alpha\centerdot\gamma}\left[{\lambda+\mu-\gamma\atop\gamma,\,\lambda-\gamma,\,\mu-\gamma}\right]\left[{\alpha\atop\lambda+\mu-\gamma}\right]$.

We now use 3.1 and 3.3 to prove 3.4 and 3.5.

Lemma 3.4. *For $A \in \Theta^{\pm}(n)$, $\lambda, \mu \in \mathbb{N}^n$ and $\delta, \gamma \in \mathbb{Z}^n$ we have*

$$0(\gamma,\mu)A(\delta,\lambda) = \sum_{\nu \in \mathbb{N}^n,\, \nu \leqslant \mu} a_\nu A(\gamma+\delta-\nu, \lambda+\mu-\nu),$$

where 0 stands for the zero matrix and

$$a_\nu = \sum_{\substack{\mathbf{j}\in\mathbb{N}^n \\ \nu-\lambda\leqslant \mathbf{j}\leqslant \nu}} v^{\mathrm{ro}(A)\centerdot(\gamma+\mu-\mathbf{j})+\lambda\centerdot(\mu-\mathbf{j})} \begin{bmatrix} \mathrm{ro}(A) \\ \mathbf{j} \end{bmatrix} \begin{bmatrix} \lambda+\mu-\nu \\ \nu-\mathbf{j},\ \lambda-\nu+\mathbf{j},\ \mu-\nu \end{bmatrix}.$$

Proof. According to (2.2.1) we have

$$0(\gamma,\mu,r)A(\delta,\lambda,r) = \sum_{\alpha\in\Lambda(n,r-\sigma(A))} v^{(\mathrm{ro}(A)+\alpha)\centerdot\gamma+\alpha\centerdot\delta} \begin{bmatrix} \mathrm{ro}(A)+\alpha \\ \mu \end{bmatrix} \begin{bmatrix} \alpha \\ \lambda \end{bmatrix} [A+\mathrm{diag}(\alpha)].$$

Furthermore by 3.3 we have

$$\begin{aligned}
\begin{bmatrix} \mathrm{ro}(A)+\alpha \\ \mu \end{bmatrix} \begin{bmatrix} \alpha \\ \lambda \end{bmatrix} &= \sum_{\mathbf{j}\in\mathbb{N}^n,\, \mathbf{j}\leqslant\mu} v^{\mathrm{ro}(A)\centerdot(\mu-\mathbf{j})-\alpha\centerdot\mathbf{j}} \begin{bmatrix} \mathrm{ro}(A) \\ \mathbf{j} \end{bmatrix} \begin{bmatrix} \alpha \\ \mu-\mathbf{j} \end{bmatrix} \begin{bmatrix} \alpha \\ \lambda \end{bmatrix} \\
&= \sum_{\substack{\mathbf{j},\beta\in\mathbb{N}^n,\, \mathbf{j}\leqslant\mu \\ \beta\leqslant\lambda,\, \beta\leqslant\mu-\mathbf{j}}} v^{(\mathrm{ro}(A)+\lambda)\centerdot(\mu-\mathbf{j})-\alpha\centerdot(\mathbf{j}+\beta)} \begin{bmatrix} \mathrm{ro}(A) \\ \mathbf{j} \end{bmatrix} \begin{bmatrix} \alpha \\ \lambda+\mu-\mathbf{j}-\beta \end{bmatrix} \\
&\qquad \times \begin{bmatrix} \lambda+\mu-\mathbf{j}-\beta \\ \beta,\ \lambda-\beta,\ \mu-\mathbf{j}-\beta \end{bmatrix}.
\end{aligned}$$

Thus we conclude that

$$\begin{aligned}
0(\gamma,\mu,r)A(\delta,\lambda,r) &= \sum_{\substack{\mathbf{j},\beta\in\mathbb{N}^n,\, \mathbf{j}\leqslant\mu \\ \beta\leqslant\lambda,\, \beta\leqslant\mu-\mathbf{j}}} v^{\mathrm{ro}(A)\centerdot(\gamma+\mu-\mathbf{j})+\lambda\centerdot(\mu-\mathbf{j})} \begin{bmatrix} \mathrm{ro}(A) \\ \mathbf{j} \end{bmatrix} \begin{bmatrix} \lambda+\mu-\mathbf{j}-\beta \\ \beta,\ \lambda-\beta,\ \mu-\mathbf{j}-\beta \end{bmatrix} \\
&\qquad \times A(\gamma+\delta-\mathbf{j}-\beta, \lambda+\mu-\mathbf{j}-\beta, r) \\
&= \sum_{\nu\in\mathbb{N}^n,\, \nu\leqslant\mu} a_\nu A(\gamma+\delta-\nu, \lambda+\mu-\nu, r).
\end{aligned}$$

The assertion follows. □

For simplicity, we set $A(\delta,\lambda,r)=0$ and $A(\delta,\lambda)=0$ if $a_{i,j}<0$ for some $i\neq j$ for $A\in M_n(\mathbb{Z})$.

Lemma 3.5. *Let $A \in \Theta^{\pm}(n)$, $\delta\in\mathbb{Z}^n$, $\lambda\in\mathbb{N}^n$, $m\in\mathbb{N}$ and $1\leqslant h\leqslant n-1$.*

(1) *For $\mathbf{t}\in\Lambda(n,m)$, $0\leqslant j\leqslant\lambda_h$, $0\leqslant k\leqslant\lambda_{h+1}$, and $0\leqslant c\leqslant\min\{t_h,j\}$, we set*

$$\begin{aligned}
\alpha^{\mathbf{t}}_{j,c,k} &= \Big(\sum_{h>u} t_u+\lambda_h-j-c\Big)\boldsymbol{e}_h + \Big(\lambda_{h+1}-k-\sum_{h+1>u} t_u\Big)\boldsymbol{e}_{h+1}, \\
\beta^{\mathbf{t}}_{j,c,k} &= (t_h+j-c-\lambda_h)\boldsymbol{e}_h + (k-\lambda_{h+1})\boldsymbol{e}_{h+1}
\end{aligned}$$

and

$$f^{\mathbf{t}}_{j,c,k}=v^{g^{\mathbf{t}}_{j,k}}\prod_{u\neq h}\overline{\left[\!\!\left[\begin{matrix}a_{h,u}+t_u\\ t_u\end{matrix}\right]\!\!\right]}\begin{bmatrix}-t_h\\ \lambda_h-j\end{bmatrix}\begin{bmatrix}t_h+j-c\\ c,\ t_h-c,\ j-c\end{bmatrix}\begin{bmatrix}t_{h+1}\\ \lambda_{h+1}-k\end{bmatrix}$$

where $g^{\mathbf{t}}_{j,k}=\sum_{j\geqslant u,\,j\neq h}a_{h,j}t_u-\sum_{j>u,\,j\neq h+1}a_{h+1,j}t_u+\sum_{u'\neq h,h+1,\,u<u'}t_ut_{u'}-t_h\delta_h+t_{h+1}\delta_{h+1}+2jt_h-kt_{h+1}$. *Then we have*

$$\begin{aligned}&(mE_{h,h+1})(\mathbf{0})A(\delta,\lambda)\\ =&\sum_{\substack{\mathbf{t}\,\in\,\Lambda(n,m)\\ 0\leqslant j\leqslant\lambda_h,\ 0\leqslant k\leqslant\lambda_{h+1}\\ 0\leqslant c\leqslant\min\{t_h,j\}}}f^{\mathbf{t}}_{j,c,k}\Big(A+\sum_{u\neq h}t_uE_{h,u}-\sum_{u\neq h+1}t_uE_{h+1,u}\Big)(\delta+\alpha^{\mathbf{t}}_{j,c,k},\lambda+\beta^{\mathbf{t}}_{j,c,k}).\end{aligned}$$

(2) *For* $\mathbf{t}\in\Lambda(n,m)$, $0\leqslant j\leqslant\lambda_{h+1}$, $0\leqslant k\leqslant\lambda_h$, *and* $0\leqslant c\leqslant\min\{t_{h+1},j\}$, *we set*

$$\begin{aligned}\widetilde{\alpha}^{\mathbf{t}}_{j,c,k}&=\Big(\sum_{h+1<u}t_u+\lambda_{h+1}-j-c\Big)\boldsymbol{e}_{h+1}+\Big(\lambda_h-k-\sum_{h<u}t_u\Big)\boldsymbol{e}_h,\\ \widetilde{\beta}^{\mathbf{t}}_{j,c,k}&=(t_{h+1}+j-c-\lambda_{h+1})\boldsymbol{e}_{h+1}+(k-\lambda_h)\boldsymbol{e}_h\end{aligned}$$

and

$$\widetilde{f}^{\mathbf{t}}_{j,c,k}=v^{\widetilde{g}^{\mathbf{t}}_{j,k}}\prod_{u\neq h+1}\overline{\left[\!\!\left[\begin{matrix}a_{h+1,u}+t_u\\ t_u\end{matrix}\right]\!\!\right]}\begin{bmatrix}-t_{h+1}\\ \lambda_{h+1}-j\end{bmatrix}\begin{bmatrix}t_{h+1}+j-c\\ c,\ t_{h+1}-c,\ j-c\end{bmatrix}\begin{bmatrix}t_h\\ \lambda_h-k\end{bmatrix}$$

where $\widetilde{g}^{\mathbf{t}}_{j,k}=\sum_{j\leqslant u,\,j\neq h+1}a_{h+1,j}t_u-\sum_{j<u,\,j\neq h}a_{h,j}t_u+\sum_{u\neq h,h+1,\,u<u'}t_ut_{u'}+t_h\delta_h-t_{h+1}\delta_{h+1}+2jt_{h+1}-kt_h$. *Then we have*

$$\begin{aligned}&(mE_{h+1,h})(\mathbf{0})A(\delta,\lambda)\\ =&\sum_{\substack{\mathbf{t}\,\in\,\Lambda(n,m)\\ 0\leqslant j\leqslant\lambda_{h+1},\ 0\leqslant k\leqslant\lambda_h\\ 0\leqslant c\leqslant\min\{t_{h+1},j\}}}\widetilde{f}^{\mathbf{t}}_{j,c,k}\Big(A-\sum_{u\neq h}t_uE_{h,u}+\sum_{u\neq h+1}t_uE_{h+1,u}\Big)(\delta+\widetilde{\alpha}^{\mathbf{t}}_{j,c,k},\lambda+\widetilde{\beta}^{\mathbf{t}}_{j,c,k}).\end{aligned}$$

Proof. For simplicity, for $A\in M_n(\mathbb{Z})$ with $\sigma(A)=r$, we set $[A]=0\in\boldsymbol{\mathcal{S}}(n,r)$ if $a_{i,j}<0$ for some i,j. According to 3.1 we have

$$\begin{aligned}&(mE_{h,h+1})(\mathbf{0},r)A(\delta,\lambda,r)\\ =&\sum_{\alpha\in\Lambda(n,r-\sigma(A))}v^{\alpha\centerdot\delta}\begin{bmatrix}\alpha\\ \lambda\end{bmatrix}[mE_{h,h+1}+\mathrm{diag}(\mathrm{ro}(A)+\alpha-m\boldsymbol{e}_{h+1})]\cdot[A+\mathrm{diag}(\alpha)]\\ =&\sum_{\substack{\alpha\in\Lambda(n,r-\sigma(A))\\ \mathbf{t}\in\Lambda(n,m)}}v^{\beta(\mathbf{t},A+\mathrm{diag}(\alpha))+\alpha\centerdot\delta-t_h\alpha_h}\prod_{u\neq h}\overline{\left[\!\!\left[\begin{matrix}a_{h,u}+t_u\\ t_u\end{matrix}\right]\!\!\right]}\begin{bmatrix}\alpha_h+t_h\\ t_h\end{bmatrix}\begin{bmatrix}\alpha\\ \lambda\end{bmatrix}\\ &\qquad\times\Big[A+\sum_{u\neq h}t_uE_{h,u}-\sum_{u\neq h+1}t_uE_{h+1,u}+\mathrm{diag}(\alpha+t_h\boldsymbol{e}_h-t_{h+1}\boldsymbol{e}_{h+1})\Big].\end{aligned}$$

Let $\nu=\alpha+t_h\boldsymbol{e}_h-t_{h+1}\boldsymbol{e}_{h+1}$. By 3.3 we have

$$\begin{aligned}\begin{bmatrix}\nu_h-t_h\\ \lambda_h\end{bmatrix}\begin{bmatrix}\nu_h\\ t_h\end{bmatrix}&=\sum_{0\leqslant j\leqslant\lambda_h}v^{\nu_h(\lambda_h-j)+jt_h}\begin{bmatrix}-t_h\\ \lambda_h-j\end{bmatrix}\left(\begin{bmatrix}\nu_h\\ t_h\end{bmatrix}\begin{bmatrix}\nu_h\\ j\end{bmatrix}\right)\\ &=\sum_{\substack{0\leqslant j\leqslant\lambda_h\\ 0\leqslant c\leqslant\min\{t_h,j\}}}v^{\nu_h(\lambda_h-j-c)+2jt_h}\begin{bmatrix}-t_h\\ \lambda_h-j\end{bmatrix}\begin{bmatrix}\nu_h\\ t_h+j-c\end{bmatrix}\begin{bmatrix}t_h+j-c\\ c,\,t_h-c,\,j-c\end{bmatrix}\end{aligned}$$

and $\left[\begin{smallmatrix}\nu_{h+1}+t_{h+1}\\ \lambda_{h+1}\end{smallmatrix}\right]=\sum_{0\leqslant k\leqslant\lambda_{h+1}}v^{\nu_{h+1}(\lambda_{h+1}-k)-kt_{h+1}}\left[\begin{smallmatrix}\nu_{h+1}\\ k\end{smallmatrix}\right]\left[\begin{smallmatrix}t_{h+1}\\ \lambda_{h+1}-k\end{smallmatrix}\right]$. This implies that

$$\begin{aligned}\begin{bmatrix}\alpha_h+t_h\\ t_h\end{bmatrix}\begin{bmatrix}\alpha\\ \lambda\end{bmatrix}=\prod_{s\neq h,h+1}\begin{bmatrix}\nu_s\\ \lambda_s\end{bmatrix}&\sum_{\substack{0\leqslant k\leqslant\lambda_{h+1},\,0\leqslant j\leqslant\lambda_h\\ 0\leqslant c\leqslant\min\{t_h,j\}}}v^{x^{\nu,\mathbf{t}}_{j,c,k}}\begin{bmatrix}-t_h\\ \lambda_h-j\end{bmatrix}\begin{bmatrix}t_h+j-c\\ c,\,t_h-c,\,j-c\end{bmatrix}\\ &\times\begin{bmatrix}t_{h+1}\\ \lambda_{h+1}-k\end{bmatrix}\begin{bmatrix}\nu_h\\ t_h+j-c\end{bmatrix}\begin{bmatrix}\nu_{h+1}\\ k\end{bmatrix}.\end{aligned}$$

where $x^{\nu,\mathbf{t}}_{j,c,k}=\nu_h(\lambda_h-j-c)+\nu_{h+1}(\lambda_{h+1}-k)+2jt_h-kt_{h+1}$. Thus

$$\begin{aligned}&(mE_{h,h+1})(\mathbf{0},r)A(\delta,\lambda,r)\\ =&\sum_{\substack{\mathbf{t}\,\in\,\Lambda(n,m)\\ 0\,\leqslant\,j\,\leqslant\,\lambda_h,\;0\,\leqslant\,k\,\leqslant\,\lambda_{h+1}\\ 0\,\leqslant\,c\,\leqslant\,\min\{t_h,\,j\}}}\prod_{u\neq h}\overline{\left[\!\!\left[\begin{matrix}a_{h,u}+t_u\\ t_u\end{matrix}\right]\!\!\right]}\begin{bmatrix}-t_h\\ \lambda_h-j\end{bmatrix}\begin{bmatrix}t_h+j-c\\ c,\,t_h-c,\,j-c\end{bmatrix}\begin{bmatrix}t_{h+1}\\ \lambda_{h+1}-k\end{bmatrix}\\ &\times\sum_{\nu\in\Lambda(n,r-\sigma(A)+t_h-t_{h+1})}v^{y^{\nu}_{j,c,k}}\prod_{s\neq h,h+1}\begin{bmatrix}\nu_s\\ \lambda_s\end{bmatrix}\begin{bmatrix}\nu_h\\ t_h+j-c\end{bmatrix}\begin{bmatrix}\nu_{h+1}\\ k\end{bmatrix}\\ &\times\Big[A+\sum_{u\neq h}t_uE_{h,u}-\sum_{u\neq h+1}t_uE_{h+1,u}+\mathrm{diag}(\nu)\Big]\\ =&\sum_{\substack{\mathbf{t}\,\in\,\Lambda(n,m)\\ 0\,\leqslant\,j\,\leqslant\,\lambda_h,\;0\,\leqslant\,k\,\leqslant\,\lambda_{h+1}\\ 0\,\leqslant\,c\,\leqslant\,\min\{t_h,\,j\}}}f^{\mathbf{t}}_{j,c,k}\Big(A+\sum_{u\neq h}t_uE_{h,u}-\sum_{u\neq h+1}t_uE_{h+1,u}\Big)(\delta+\alpha^{\mathbf{t}}_{j,c,k},\lambda+\beta^{\mathbf{t}}_{j,c,k}).\end{aligned}$$

where $y^{\nu,\mathbf{t}}_{j,c,k}=\beta(\mathbf{t},A+\mathrm{diag}(\alpha))+\alpha\centerdot\delta-t_h\alpha_h+x^{\nu,\mathbf{t}}_{j,c,k}=g^{\mathbf{t}}_{j,k}+\nu\centerdot(\delta+\alpha^{\mathbf{t}}_{j,c,k})$. The assertion (1) follows. The assertion (2) can be proved in a way similar to the proof of (1). $\square$

4. Realization of $U(n)$ and $\overline{U_k(n)}$

We shall denote by $\mathcal{V}(n)$ the $\mathcal{Z}$-submodule of $\prod_{r\geqslant0}\boldsymbol{\mathcal{S}}(n,r)$ spanned by $\{A(\delta,\lambda)\mid A\in\Theta^{\pm}(n),\,\delta\in\mathbb{Z}^n,\,\lambda\in\mathbb{N}^n\}$. Let $\mathcal{V}^0(n)$ be the $\mathcal{Z}$-subalgebra of $\prod_{r\geqslant0}\boldsymbol{\mathcal{S}}(n,r)$ generated by $0(\pm\boldsymbol{e}_i)$ and $0(0,t\boldsymbol{e}_i)$ for $1\leqslant i\leqslant n$ and $t\in\mathbb{N}$, where $\boldsymbol{e}_i=(0,\cdots,0,\underset{i}{1},0\cdots,0)\in\mathbb{N}^n$.

Lemma 4.1. *The set $\{0(\delta,\lambda)\mid\delta,\lambda\in\mathbb{N}^n,\,\delta_i\in\{0,1\},\forall i\}$ forms a $\mathcal{Z}$-basis for $\mathcal{V}^0(n)$.*

Proof. Let $\boldsymbol{\mathcal{V}}^0(n)$ be the $\mathbb{Q}(v)$-subalgebra of $\prod_{r\geqslant 0}\boldsymbol{\mathcal{S}}(n,r)$ generated by $0(\pm\boldsymbol{e}_i)$ for $1\leqslant i\leqslant n$. Since the set $\{0(\mathbf{j})\mid \mathbf{j}\in\mathbb{Z}^n\}$ forms a $\mathbb{Q}(v)$-basis for $\boldsymbol{\mathcal{V}}^0(n)$ we conclude that $\boldsymbol{\mathcal{V}}^0(n)$ is isomorphic to $\mathbf{U}^0(n)$, where $\mathbf{U}^0(n)$ is the $\mathbb{Q}(v)$-subalgebra of $\mathbf{U}(n)$ generated by $K_i^{\pm 1}$ for $1\leqslant i\leqslant n$. Now the assertion follows from [13, 4.5]. □

We now describe several $\mathcal{Z}$-bases for $\mathcal{V}(n)$ as follows.

Lemma 4.2. *Each of the following set forms a $\mathcal{Z}$-basis for $\mathcal{V}(n)$:*

(1) $\mathfrak{B}_1=\{0(\delta,\lambda)A(\mathbf{0})\mid A\in\Theta^\pm(n),\ \delta,\lambda\in\mathbb{N}^n,\ \delta_i\in\{0,1\},\forall i\}$;
(2) $\mathfrak{B}_2=\{A(\mathbf{0})0(\delta,\lambda)\mid A\in\Theta^\pm(n),\ \delta,\lambda\in\mathbb{N}^n,\ \delta_i\in\{0,1\},\forall i\}$;
(3) $\mathfrak{B}_3=\{A(\delta,\lambda)\mid A\in\Theta^\pm(n),\ \delta,\lambda\in\mathbb{N}^n,\ \delta_i\in\{0,1\},\forall i\}$.

Proof. According to 3.4 we have

$$0(\delta,\lambda)A(\mathbf{0})=v^{\mathrm{ro}(A)\centerdot(\delta+\lambda)}A(\delta,\lambda)+\sum_{\mathbf{j}\in\mathbb{N}^n,\,\mathbf{0}<\mathbf{j}\leqslant\lambda}v^{\mathrm{ro}(A)\centerdot(\delta+\lambda-\mathbf{j})}\begin{bmatrix}\mathrm{ro}(A)\\ \mathbf{j}\end{bmatrix}A(\delta-\mathbf{j},\lambda-\mathbf{j}).$$

It follows that $\mathcal{V}(n)$ is spanned by $\{0(\delta,\lambda)A(\mathbf{0})\mid A\in\Theta^\pm(n),\ \delta\in\mathbb{Z}^n,\ \lambda\in\mathbb{N}^n\}$. Thus by 4.1 we have $\mathcal{V}(n)=\mathrm{span}_{\mathcal{Z}}\,\mathfrak{B}_1$. Since the set $\{0(\mathbf{j})A(\mathbf{0})\mid A\in\Theta^\pm(n),\ \mathbf{j}\in\mathbb{Z}^n\}$ is linearly independent, by 4.1 we conclude that the set $\mathfrak{B}_1$ is linearly independent. Hence the set $\mathfrak{B}_1$ forms a $\mathcal{Z}$-basis for $\mathcal{V}(n)$. Similarly, the set $\mathfrak{B}_2$ forms a $\mathcal{Z}$-basis for $\mathcal{V}(n)$. It remains to prove that the set $\mathfrak{B}_3$ forms a $\mathcal{Z}$-basis for $\mathcal{V}(n)$. For $\lambda\in\mathbb{N}^n$ and $\mu,\delta\in\mathbb{Z}^n$ we have

$$v^{\delta_i\mu_i}\begin{bmatrix}\mu_i\\ \lambda_i\end{bmatrix}=v^{\lambda_i}(v^{\lambda_i+1}-v^{-\lambda_i-1})v^{(\delta_i-1)\mu_i}\begin{bmatrix}\mu_i\\ \lambda_i+1\end{bmatrix}+v^{2\lambda_i+(\delta_i-2)\mu_i}\begin{bmatrix}\mu_i\\ \lambda_i\end{bmatrix}.$$

It follows that

$$\begin{aligned}A(\delta,\lambda)&=v^{\lambda_i}(v^{\lambda_i+1}-v^{-\lambda_i-1})A(\delta-\boldsymbol{e}_i,\lambda+\boldsymbol{e}_i)+v^{2\lambda_i}A(\delta-2\boldsymbol{e}_i,\lambda)\\&=-v^{-\lambda_i}(v^{\lambda_i+1}-v^{-\lambda_i-1})A(\delta+\boldsymbol{e}_i,\lambda+\boldsymbol{e}_i)+v^{-2\lambda_i}A(\delta+2\boldsymbol{e}_i,\lambda)\end{aligned}$$

for $1\leqslant i\leqslant n$, $\lambda\in\mathbb{N}^n$ and $\delta\in\mathbb{Z}^n$. This shows that $\mathcal{V}(n)$ is spanned by $\mathfrak{B}_3$. Assume

$$\sum_{\substack{A\in\Theta^\pm(n),\,\lambda,\delta\in\mathbb{N}^n\\ \delta_i\in\{0,1\},\forall i}}f_{A,\delta,\lambda}A(\delta,\lambda)=0$$

where $f_{A,\delta,\lambda}\in\mathbb{Q}(v)$. Then

$$\sum_{\substack{A\in\Theta^\pm(n)\\ \mu\in\Lambda(n,r-\sigma(A))}}\Bigg(\sum_{\substack{\lambda,\delta\in\mathbb{N}^n\\ \delta_i\in\{0,1\},\forall i}}f_{A,\delta,\lambda}v^{\mu\centerdot\delta}\begin{bmatrix}\mu\\ \lambda\end{bmatrix}\Bigg)[A+\mathrm{diag}(\mu)]=\sum_{\substack{A\in\Theta^\pm(n),\,\lambda,\delta\in\mathbb{N}^n\\ \delta_i\in\{0,1\},\forall i}}f_{A,\delta,\lambda}A(\delta,\lambda,r)=0.$$

This implies that

$$\sum_{\substack{\lambda,\delta\in\mathbb{N}^n\\ \delta_i\in\{0,1\},\forall i}}f_{A,\delta,\lambda}v^{\mu\centerdot\delta}\begin{bmatrix}\mu\\ \lambda\end{bmatrix}=0$$

for $A \in \Theta^{\pm}(n)$ and $\mu \in \mathbb{N}^n$. It follows that

$$\sum_{\substack{\lambda,\delta\in\mathbb{N}^n\\ \delta_i\in\{0,1\},\forall i}} f_{A,\delta,\lambda} 0(\delta,\lambda,r) = 0$$

for $r \geqslant 0$ and $A \in \Theta^{\pm}(n)$. Thus by 4.1 we conclude that $f_{A,\delta,\lambda} = 0$ for all A, δ, λ. This shows that the set $\mathfrak{B}_3$ is linearly independent and hence the set $\mathfrak{B}_3$ forms a $\mathcal{Z}$-basis for $\mathcal{V}(n)$. □

We now use 3.4 and 3.5 to prove that $\mathcal{V}(n)$ is a $\mathcal{Z}$-subalgebra of $\prod_{r\geqslant 0} \boldsymbol{\mathcal{S}}(n,r)$.

Proposition 4.3. *$\mathcal{V}(n)$ is a $\mathcal{Z}$-subalgebra of $\prod_{r\geqslant 0} \boldsymbol{\mathcal{S}}(n,r)$. Furthermore the elements $(mE_{h,h+1})(\mathbf{0})$, $(mE_{h+1,h})(\mathbf{0})$ and $0(\delta,\lambda)$ (for $m \in \mathbb{N}$, $1 \leqslant h \leqslant n-1$, $\delta \in \mathbb{Z}^n$ and $\lambda \in \mathbb{N}^n$) generate $\mathcal{V}(n)$ as a $\mathcal{Z}$-algebra.*

Proof. Let $\mathcal{V}(n)_1$ be the $\mathcal{Z}$-subalgebra of $\prod_{r\geqslant 0} \boldsymbol{\mathcal{S}}(n,r)$ generated by $(mE_{h,h+1})(\mathbf{0})$, $(mE_{h+1,h})(\mathbf{0})$ and $0(\delta,\lambda)$ for $m \in \mathbb{N}$, $1 \leqslant h \leqslant n-1$, $\delta \in \mathbb{Z}^n$ and $\lambda \in \mathbb{N}^n$. From 3.4 and 3.5 we see that

$$\mathcal{V}(n)_1 \subseteq \mathcal{V}(n)_1\mathcal{V}(n) \subseteq \mathcal{V}(n). \tag{4.3.1}$$

So by 4.2 it is enough to prove $A(\mathbf{0})0(\delta,\lambda) \in \mathcal{V}(n)_1$ for $A \in \Theta^{\pm}(n)$, $\delta, \lambda \in \mathbb{N}^n$ with $\delta_i \in \{0,1\}$ $(1 \leqslant i \leqslant n)$. We shall prove this by induction on $\|A\|$, where

$$\|A\| = \sum_{r<s} \frac{(s-r)(s-r+1)}{2} a_{rs} + \sum_{r>s} \frac{(r-s)(r-s+1)}{2} a_{rs} \in \mathbb{N}.$$

If $\|A\| = 0$, then $A(\mathbf{0})0(\delta,\lambda) = 0(\delta,\lambda) \in \mathcal{V}(n)_1$. Now we assume that $\|A\| > 0$ and our statement is true for A' with $\|A'\| < \|A\|$. According to 2.3, for $A \in \Theta^{\pm}(n)$, we have

$$\prod_{1\leqslant i\leqslant h<j\leqslant n} (a_{i,j}E_{h,h+1})(\mathbf{0}) \cdot \prod_{1\leqslant j\leqslant h<i\leqslant n} (a_{i,j}E_{h+1,h})(\mathbf{0}) = A(\mathbf{0}) + f$$

where f is the $\mathbb{Q}(v)$-linear combination of $B(\mathbf{0})0(\mathbf{j})$ with $B \in \Theta^{\pm}(n)$, $B \prec A$ and $\mathbf{j} \in \mathbb{Z}^n$. It follows that

$$\prod_{1\leqslant i\leqslant h<j\leqslant n} (a_{i,j}E_{h,h+1})(\mathbf{0}) \cdot \prod_{1\leqslant j\leqslant h<i\leqslant n} (a_{i,j}E_{h+1,h})(\mathbf{0}) \cdot 0(\delta,\lambda) = A(\mathbf{0})0(\delta,\lambda) + g \tag{4.3.2}$$

for $\delta, \lambda \in \mathbb{N}^n$ with $\delta_i \in \{0,1\}$ $(1 \leqslant i \leqslant n)$, where $g = f \cdot 0(\delta,\lambda)$. From (4.3.1), 4.1 and 4.2 we see that g must be a $\mathcal{Z}$-linear combination of $B(\mathbf{0})0(\gamma,\mu)$ with $B \in \Theta^{\pm}(n)$, $B \prec A$, $\gamma, \mu \in \mathbb{N}^n$ and $\gamma_i \in \{0,1\}$ for $1 \leqslant i \leqslant n$. Note that if $B \in \Theta^{\pm}(n)$ satisfy $B \prec A$, then $\|B\| < \|A\|$ (see the proof of [1, 4.2]). Thus by induction we conclude that $g \in \mathcal{V}(n)_1$ and hence $A(\mathbf{0})0(\delta,\lambda) \in \mathcal{V}(n)_1$. The assertion follows. □

Theorem 4.4. *There is a $\mathcal{Z}$-algebra isomorphism $\zeta : U(n) \to \mathcal{V}(n)$ satisfying $E_h^{(m)} \mapsto (mE_{h,h+1})(\mathbf{0})$, $F_h^{(m)} \mapsto (mE_{h+1,h})(\mathbf{0})$ and $\prod_{1\leqslant i\leqslant n} K_i^{\delta_i} \left[{K_i;0 \atop \lambda_i} \right] \mapsto 0(\delta,\lambda)$ for $m \in \mathbb{N}$, $1 \leqslant h \leqslant n-1$, $\delta \in \mathbb{Z}^n$ and $\lambda \in \mathbb{N}^n$.*

Proof. The maps $\zeta_r : \mathbf{U}(n) \to \boldsymbol{\mathcal{S}}(n,r)$ induce an algebra homomorphism

$$\zeta : \mathbf{U}(n) \to \prod_{r\geqslant 0} \boldsymbol{\mathcal{S}}(n,r)$$

satisfying $\zeta(x) = (\zeta_r(x))_{r\geqslant 0}$ for $x \in \mathbf{U}(n)$. From 4.3 we see that $\zeta(U(n)) = \mathcal{V}(n)$. Furthermore by 2.2, 4.2 and (4.3.2), we conclude that ζ is injective. □

Remark 4.5. (1) Let $\boldsymbol{\mathcal{V}}(n)$ be the $\mathbb{Q}(v)$-subspace of $\prod_{r\geqslant 0} \boldsymbol{\mathcal{S}}(n,r)$ spanned by $\{A(\delta) \mid A \in \Theta^\pm(n),\, \delta \in \mathbb{Z}^n\}$. Then $\boldsymbol{\mathcal{V}}(n) \cong \mathcal{V}(n) \otimes_{\mathcal{Z}} \mathbb{Q}(v)$. According to 4.3 and 4.4 we conclude that $\boldsymbol{\mathcal{V}}(n)$ is a $\mathbb{Q}(v)$-subalgebra of $\prod_{r\geqslant 0} \boldsymbol{\mathcal{S}}(n,r)$ and $\mathbf{U}(n) \cong \boldsymbol{\mathcal{V}}(n)$.

(2) Note that for $A \in \Theta^\pm(n)$ and $\lambda \in \Lambda(n, r-\sigma(A))$ we have $A(\mathbf{0}, \lambda, r) = [A + \mathrm{diag}(\lambda)]$. Thus from 4.4 we see that $\zeta_r(U(n)) = \mathcal{S}(n,r)$, which has been proved in [6, 3.4].

We now use q-Schur algebras over k to realize quantum $\mathfrak{gl}_n$ over k, where k is a field containing an l-th primitive root ε of 1 with $l \geqslant 1$ odd. Specializing v to ε, k will be viewed as a $\mathcal{Z}$-module. For $\mu \in \mathbb{Z}^n$ and $\lambda \in \mathbb{N}^n$ we shall denote the image of $\left[\begin{smallmatrix}\mu\\ \lambda\end{smallmatrix}\right]$ in k by $\left[\begin{smallmatrix}\mu\\ \lambda\end{smallmatrix}\right]_\varepsilon$. Let $U_k(n) = U(n) \otimes_{\mathcal{Z}} k$ and $\mathcal{S}_k(n,r) = \mathcal{S}(n,r) \otimes_{\mathcal{Z}} k$. By restriction, the map $\zeta_r : \mathbf{U}(n) \to \boldsymbol{\mathcal{S}}(n,r)$ induces an algebra homomorphism $\zeta_r : U(n) \to \mathcal{S}(n,r)$. By tensoring with the field k, we get an algebra homomorphism

$$\zeta_{r,k} := \zeta_r \otimes id : U_k(n) \to \mathcal{S}_k(n,r).$$

Let

$$\overline{U_k(n)} = U_k(n)/\langle K_i^l - 1 \mid 1 \leqslant i \leqslant n-1\rangle.$$

Since $\zeta_{r,k}(K_i^l) = 1$, $\zeta_{r,k}$ induces an algebra homomorphism

$$\bar{\zeta}_{r,k} : \overline{U_k(n)} \to \mathcal{S}_k(n,r)$$

satisfying $\bar{\zeta}_{r,k}(\bar{x}) = \zeta_{r,k}(x)$ for $x \in U_k(n)$. The maps $\bar{\zeta}_{r,k}$ induce an algebra homomorphism

$$\bar{\zeta}_k := \prod_{r\geqslant 0} \bar{\zeta}_{r,k} : \overline{U_k(n)} \to \prod_{r\geqslant 0} \mathcal{S}_k(n,r)$$

satisfying $\bar{\zeta}_k(x) = (\bar{\zeta}_{r,k}(\bar{x}))_{r\geqslant 0}$ for $\bar{x} \in \overline{U_k(n)}$. For $A \in \Theta(n,r)$ we let $[A]_\varepsilon = [A] \otimes 1 \in \mathcal{S}_k(n,r)$. Similarly, for $A \in \Theta^\pm(n)$, $\delta \in \mathbb{Z}^n$ and $\lambda \in \mathbb{N}^n$, let $A(\delta,\lambda,r)_\varepsilon = A(\delta,\lambda,r) \otimes 1 \in \mathcal{S}_k(n,r)$, $A(\delta,\lambda)_\varepsilon = (A(\delta,\lambda,r)_\varepsilon)_{r\geqslant 0} \in \prod_{r\geqslant 0} \mathcal{S}_k(n,r)$ and $A(\delta)_\varepsilon = A(\delta, \mathbf{0})_\varepsilon$. From 4.2 and 4.4 we see that

$$\bar{\zeta}_k(\overline{U_k(n)}) = \mathrm{span}_k\{A(\delta,\lambda)_\varepsilon \mid A \in \Theta^\pm(n),\, \delta, \lambda \in \mathbb{N}^n,\, \delta_i \in \{0,1\}, \forall i\}.$$

Theorem 4.6. *The algebra homomorphism $\bar{\zeta}_k$ is injective. Furthermore, the set*

$$\mathfrak{B}_k := \{A(\mathbf{0})_\varepsilon 0(-\lambda,\lambda)_\varepsilon \mid A \in \Theta^\pm(n),\, \lambda \in \mathbb{N}^n\}$$

forms a k-basis of $\bar{\zeta}_k(\overline{U_k(n)})$.

Proof. We will identify $U(n)$ with $\mathcal{V}(n)$ via the map ζ defined in 4.4. From 4.2 and [13, 6.4(b)], we see that the set

$$\{A(\mathbf{0})0(l\delta)0(-\lambda,\lambda)\otimes 1 \mid A\in\Theta^{\pm}(n),\ \lambda,\delta\in\mathbb{N}^n,\ \delta_i\in\{0,1\},\forall i\}$$

forms a k-basis for $U_k(n)$. It follows that $\bar{\zeta}_k(\overline{U_k(n)})$ is spanned by the set $\mathfrak{B}_k$. Thus it is enough to prove that the set $\mathfrak{B}_k$ is linearly independent.

Assume

$$\sum_{A\in\Theta^{\pm}(n),\,\lambda\in\mathbb{N}^n} f_{A,\lambda}A(\mathbf{0})_\varepsilon 0(-\lambda,\lambda)_\varepsilon = 0$$

where $f_{A,\lambda}\in k$. Then for any $r\geqslant 0$

$$\sum_{\substack{A\in\Theta^{\pm}(n)\\ \mu\in\Lambda(n,r-\sigma(A))}}\Big(\sum_{\lambda\in\mathbb{N}^n} f_{A,\lambda}\varepsilon^{-\lambda\centerdot(\mu+\mathrm{co}(A))}\begin{bmatrix}\mu+\mathrm{co}(A)\\ \lambda\end{bmatrix}_\varepsilon\Big)[A+\mathrm{diag}(\mu)]_\varepsilon = 0.$$

It follows that for any $A\in\Theta^{\pm}(n)$, $\mu\in\Lambda(n,r-\sigma(A))$ with $r\geqslant\sigma(A)$, we have

$$\sum_{\lambda\in\mathbb{N}^n} f_{A,\lambda}\varepsilon^{-\lambda\centerdot(\mu+\mathrm{co}(A))}\begin{bmatrix}\mu+\mathrm{co}(A)\\ \lambda\end{bmatrix}_\varepsilon = 0. \tag{4.6.1}$$

We claim that for $A\in\Theta^{\pm}(n)$ and $\mu,\alpha\in\mathbb{N}^n$ we have

$$\sum_{\lambda\in\mathbb{N}^n,\,\lambda\geqslant\alpha} f_{A,\lambda}\varepsilon^{-\lambda\centerdot(\mu+\mathrm{co}(A))}\begin{bmatrix}\mu+\mathrm{co}(A)\\ \lambda-\alpha\end{bmatrix}_\varepsilon = 0. \tag{4.6.2}$$

We apply induction on $\sigma(\alpha)$. For $A\in\Theta^{\pm}(n)$ and $\mu,\alpha\in\mathbb{N}^n$, we denote

$$g_{A,\alpha,\mu} = \sum_{\lambda\in\mathbb{N}^n,\,\lambda\geqslant\alpha} f_{A,\lambda}\varepsilon^{-\lambda\centerdot(\mu+\mathrm{co}(A))}\begin{bmatrix}\mu+\mathrm{co}(A)\\ \lambda-\alpha\end{bmatrix}_\varepsilon.$$

If $\sigma(\alpha)=0$ then the claim follows from (4.6.1). Now we assume $\sigma(\alpha)>0$. There exist $\beta\in\mathbb{N}^n$ such that $\alpha=\beta+\boldsymbol{e}_i$. According to 3.3(1), for $\lambda\in\mathbb{N}^n$ with $\lambda\geqslant\beta$ we have

$$\begin{bmatrix}\mu+\boldsymbol{e}_i+\mathrm{co}(A)\\ \lambda-\beta\end{bmatrix}_\varepsilon = \varepsilon^{\lambda_i-\beta_i}\begin{bmatrix}\mu+\mathrm{co}(A)\\ \lambda-\beta\end{bmatrix}_\varepsilon + \varepsilon^{\lambda_i-\beta_i-1-\mu_i-\sum_{1\leqslant k\leqslant n}a_{k,i}}\begin{bmatrix}\mu+\mathrm{co}(A)\\ \lambda-\beta-\boldsymbol{e}_i\end{bmatrix}_\varepsilon.$$

Thus by the induction hypothesis we conclude that

$$0 = g_{A,\beta,\mu+\boldsymbol{e}_i} = \varepsilon^{-\beta_i}g_{A,\beta,\mu} + \varepsilon^{-\beta_i-1-\sum_{1\leqslant k\leqslant n}a_{k,i}-\mu_i}g_{A,\alpha,\mu} = \varepsilon^{-\beta_i-1-\sum_{1\leqslant k\leqslant n}a_{k,i}-\mu_i}g_{A,\alpha,\mu}.$$

for $A\in\Theta^{\pm}(n)$ and $\mu\in\mathbb{N}^n$. It follows that $g_{A,\alpha,\mu}=0$ for $A\in\Theta^{\pm}(n)$ and $\mu\in\mathbb{N}^n$, proving (4.6.2).

Let $\mathcal{X}=\{\lambda\in\mathbb{N}^n \mid f_{A,\lambda}\neq 0 \text{ for some } A\in\Theta^{\pm}(n)\}$. If $\mathcal{X}\neq\emptyset$, we may choose a maximal element ν in $\mathcal{X}$ with respect to $\leqslant$. Then by (4.6.2) we have

$$f_{A,\nu} = \varepsilon^{\nu\centerdot(\mu+\mathrm{co}(A))}\sum_{\lambda\in\mathbb{N}^n,\,\lambda\geqslant\nu} f_{A,\lambda}\varepsilon^{-\lambda\centerdot(\mu+\mathrm{co}(A))}\begin{bmatrix}\mu+\mathrm{co}(A)\\ \lambda-\nu\end{bmatrix}_\varepsilon = 0.$$

for $A \in \Theta^{\pm}(n)$. This is a contradiction. Thus $f_{A,\lambda} = 0$ for all $A \in \Theta^{\pm}(n)$ and $\lambda \in \mathbb{N}^n$. The assertion follows. □

Remark 4.7. (1) Let $\mathcal{U}(\mathfrak{gl}_n)$ be the universal enveloping algebra of $\mathfrak{gl}_n$ and let $\mathcal{U}_{\mathbb{Z}}(\mathfrak{gl}_n)$ be the Kostant $\mathbb{Z}$-form of $\mathcal{U}(\mathfrak{gl}_n)$. Let $\mathcal{S}_{\mathbb{Q}}(n,r) = \mathcal{S}(n,r) \otimes_{\mathcal{Z}} \mathbb{Q}$, $U_{\mathbb{Z}}(n) = U(n) \otimes_{\mathcal{Z}} \mathbb{Z}$, where $\mathbb{Z}$ and $\mathbb{Q}$ are regarded as $\mathcal{Z}$-modules by specializing v to 1. Let $\mathcal{W}_{\mathbb{Z}}(n)$ be the $\mathbb{Z}$-submodule of $\prod_{r\geqslant 0} \mathcal{S}_{\mathbb{Q}}(n,r)$ spanned by the set $\{A(\mathbf{0},\lambda)_1 \mid A \in \Theta^{\pm}_{\vartriangle}(n),\ \lambda \in \mathbb{N}^n\}$. According to [13, 6.7(c)], 4.4 and 4.6 we conclude that $\mathcal{W}_{\mathbb{Z}}(n)$ is a $\mathbb{Z}$ algebra and $\mathcal{U}_{\mathbb{Z}}(\mathfrak{gl}_n) \cong U_{\mathbb{Z}}(n)/\langle K_i - 1 \mid 1 \leqslant i \leqslant n\rangle \cong \mathcal{W}_{\mathbb{Z}}(n)$.

(2) Assume $\varepsilon = 1 \in k$. Then $l = 1$ and $\mathcal{S}_k(n,r)$ is the Schur algebra over k. Let $\mathcal{W}_k(n)$ be the k-subspace of $\prod_{r\geqslant 0} \mathcal{S}_k(n,r)$ spanned by the set $\{A(\mathbf{0},\lambda)_1 \mid A \in \Theta^{\pm}_{\vartriangle}(n),\ \lambda \in \mathbb{N}^n\}$. From [13, 6.7(c)] and 4.6 we see that $\mathcal{W}_k(n)$ is a k-algebra and $\mathcal{U}_{\mathbb{Z}}(\mathfrak{gl}_n) \otimes_{\mathbb{Z}} k \cong \overline{U_k(n)} \cong \mathcal{W}_k(n)$.

We end this paper with a conjecture on affine q-Schur algebras. Let $\Theta_{\vartriangle}(n)$ be the set of all $\mathbb{Z} \times \mathbb{Z}$ matrices $A = (a_{i,j})_{i,j\in\mathbb{Z}}$ with $a_{i,j} \in \mathbb{N}$ such that

(a) $a_{i,j} = a_{i+n,j+n}$ for $i, j \in \mathbb{Z}$, and
(b) for every $i \in \mathbb{Z}$, both sets $\{j \in \mathbb{Z} \mid a_{i,j} \neq 0\}$ and $\{j \in \mathbb{Z} \mid a_{j,i} \neq 0\}$ are finite.

Let $\mathbb{Z}^n_{\vartriangle} = \{(\lambda_i)_{i\in\mathbb{Z}} \mid \lambda_i \in \mathbb{Z},\ \lambda_i = \lambda_{i-n} \text{ for } i \in \mathbb{Z}\}$ and $\mathbb{N}^n_{\vartriangle} = \{(\lambda_i)_{i\in\mathbb{Z}} \in \mathbb{Z}^n_{\vartriangle} \mid \lambda_i \geqslant 0\}$. For $r \in \mathbb{N}$ let $\Theta_{\vartriangle}(n,r) = \{A \in \Theta_{\vartriangle}(n) \mid \sigma(A) = r\}$ and $\Lambda_{\vartriangle}(n,r) = \{\lambda \in \mathbb{N}^n_{\vartriangle} \mid \sigma(\lambda) = r\}$ where $\sigma(\lambda) = \sum_{1\leqslant i\leqslant n} \lambda_i$ and $\sigma(A) = \sum_{1\leqslant i\leqslant n,\, j\in\mathbb{Z}} a_{i,j}$. For $\lambda \in \Lambda_{\vartriangle}(n,r)$, let $\mathrm{diag}(\lambda) = (\delta_{i,j}\lambda_i)_{i,j\in\mathbb{Z}} \in \Theta_{\vartriangle}(n,r)$.

Let $\mathcal{S}_{\vartriangle}(n,r)$ be the affine q-Schur algebra over $\mathcal{Z}$. It has a normalized $\mathcal{Z}$-basis $\{[A] \mid A \in \Theta_{\vartriangle}(n,r)\}$ (see [15, 1.9]). We put $\boldsymbol{\mathcal{S}}_{\vartriangle}(n,r) = \mathcal{S}_{\vartriangle}(n,r) \otimes_{\mathcal{Z}} \mathbb{Q}(v)$.

Let $\Theta^{\pm}_{\vartriangle}(n) = \{A \in \Theta_{\vartriangle}(n) \mid a_{i,i} = 0 \text{ for all } i\}$. For $A \in \Theta^{\pm}_{\vartriangle}(n)$, $\delta \in \mathbb{Z}^n_{\vartriangle}$ and $\lambda \in \mathbb{N}^n_{\vartriangle}$ let

$$A(\delta,\lambda,r) = \sum_{\mu\in\Lambda_{\vartriangle}(n,r-\sigma(A))} v^{\mu\centerdot\delta} \begin{bmatrix}\mu\\ \lambda\end{bmatrix} [A + \mathrm{diag}(\mu)] \in \boldsymbol{\mathcal{S}}_{\vartriangle}(n,r)$$

$$A(\delta,\lambda) = (A(\delta,\lambda,r))_{r\geqslant 0} \in \prod_{r\geqslant 0} \boldsymbol{\mathcal{S}}_{\vartriangle}(n,r)$$

where $\mu \centerdot \delta = \sum_{1\leqslant i\leqslant n} \mu_i\delta_i$ and $\left[\begin{smallmatrix}\mu\\ \lambda\end{smallmatrix}\right] = \left[\begin{smallmatrix}\mu_1\\ \lambda_1\end{smallmatrix}\right] \cdots \left[\begin{smallmatrix}\mu_n\\ \lambda_n\end{smallmatrix}\right]$. Let $A(\delta) = A(\delta,\mathbf{0})$, where $\mathbf{0} = (\cdots,0,\cdots,0,\cdots) \in \mathbb{N}^n_{\vartriangle}$.

We shall denote by $\mathcal{V}_{\vartriangle}(n)$ the $\mathcal{Z}$-submodule of $\prod_{r\geqslant 0} \boldsymbol{\mathcal{S}}_{\vartriangle}(n,r)$ spanned by $\{A(\delta,\lambda) \mid A \in \Theta^{\pm}_{\vartriangle}(n),\ \delta \in \mathbb{Z}^n_{\vartriangle},\ \lambda \in \mathbb{N}^n_{\vartriangle}\}$.

Lemma 4.8. *Each of the following set forms a $\mathcal{Z}$-basis for $\mathcal{V}_{\vartriangle}(n)$:*

(1) $\{0(\delta,\lambda)A(\mathbf{0}) \mid A \in \Theta^{\pm}_{\vartriangle}(n),\ \delta,\lambda \in \mathbb{N}^n_{\vartriangle},\ \delta_i \in \{0,1\}, \forall i\}$;
(2) $\{A(\mathbf{0})0(\delta,\lambda) \mid A \in \Theta^{\pm}_{\vartriangle}(n),\ \delta,\lambda \in \mathbb{N}^n_{\vartriangle},\ \delta_i \in \{0,1\}, \forall i\}$;
(3) $\{A(\delta,\lambda) \mid A \in \Theta^{\pm}_{\vartriangle}(n),\ \delta,\lambda \in \mathbb{N}^n_{\vartriangle},\ \delta_i \in \{0,1\}, \forall i\}$.

Proof. The assertion can be proved in a way similar to the proof of 4.2. □

According to 4.3, $\mathcal{V}(n)$ is a $\mathcal{Z}$-subalgebra of $\prod_{r\geqslant 0}\boldsymbol{\mathcal{S}}(n,r)$. Thus, it is natural to formulate the following conjecture.

Conjecture 4.9. *$\mathcal{V}_\triangle(n)$ is a $\mathcal{Z}$-subalgebra of $\prod_{r\geqslant 0}\boldsymbol{\mathcal{S}}_\triangle(n,r)$.*

Remarks 4.10. (1) According to [8], Conjecture 4.9 is true in the classical ($v=1$) case.

(2) Let $\boldsymbol{\mathcal{V}}_\triangle(n)$ be the $\mathbb{Q}(v)$-subspace of $\prod_{r\geqslant 0}\boldsymbol{\mathcal{S}}_\triangle(n,r)$ spanned by all $A(\delta)$ for $A\in\Theta_\triangle^\pm(n)$ and $\delta\in\mathbb{Z}^n$. It is conjectured in [7, 5.5(2)] that $\boldsymbol{\mathcal{V}}_\triangle(n)$ is a $\mathbb{Q}(v)$-subalgebra of $\prod_{r\geqslant 0}\boldsymbol{\mathcal{S}}_\triangle(n,r)$. From 4.1 and 4.8, we see that $\boldsymbol{\mathcal{V}}_\triangle(n)\cong\mathcal{V}_\triangle(n)\otimes\mathbb{Q}(v)$. Thus if Conjecture 4.9 is true, then we conclude that $\boldsymbol{\mathcal{V}}_\triangle(n)$ is a $\mathbb{Q}(v)$-subalgebra of $\prod_{r\geqslant 0}\boldsymbol{\mathcal{S}}_\triangle(n,r)$.

(3) If Conjecture 4.9 is true, then by [2, 3.7.3] we conclude that the conjecture formulated in [2, 3.8.6] is true and $\mathcal{V}_\triangle(n)$ is isomorphic to $\widetilde{\mathfrak{D}}_\triangle(n)$, where $\widetilde{\mathfrak{D}}_\triangle(n)$ is a certain $\mathcal{Z}$-module defined in [2, (3.8.1.1)].

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Department of Mathematics, Tongji University, Shanghai, 200092, China.
E-mail address: q.fu@tongji.edu.cn